1 chapter 1

Definition 1.1 (Homeomorphism)**.** *A bijection $f : X \to Y$ is called a homeomorphism between two topological spaces X and Y if f and f^{-1} are continuous mappings.*

Definition 1.2 (Connected Topological Space)**.** *A topological space is connected if it is not the union of two non-empty, open, disjoint subspace topologies.*

2 chapter 2

Definition 2.1 (Filter)**.** *Given a set X, a filter $\mathcal{F}$ on X is a nonempty family of nonempty subsets of X such that*

- *if $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$*
- *if $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$*

Examples:

- Principal filter: given a subset Y of X, the family of subsets containing Y is a filter.
- Frechet filter is the family of all cofinite subsets of X.
- Free filter: $\mathcal{F}$ is a free filter if $\bigcap_{A \in \mathcal{F}} A = \emptyset$.

Remarks: A principal filter is not free; A filter is free iff it contains the Frechet filter.

Proof. All subsets in a principal filter contain a common subset Y in X, so their intersection is not empty. The intersection of all subsets in a Frechet filter is empty. Consider the subsets A such that X
$A = x$ where $x \in X$, they are contained in the Frechet filter, yet their intersection is empty: $\bigcup_A A^c = X$ where A is as described above. On the other hand, if a filter does not contain a Frechet filter, then there is a cofinite subset $B \subset X$ and $B \notin \mathcal{F}$. $\forall A \in \mathcal{F}$, $A \not\subset B$, because otherwise, by the property of filters, $B \in \mathcal{F}$. $B^c \not\subset A^c$, so the union of all A^c cannot cover X, namely, $\bigcap_{A \in \mathcal{F}} A \neq \emptyset$ and $\mathcal{F}$ is not a free filter. $(\bigcap_{A \in \mathcal{F}} A)^c = \bigcup_{A \in \mathcal{F}} A^c$ □

Remark: given $Y \subset X$, the principal filter $\mathcal{F}_Y$ is an ultra filter iff $|Y| = 1$.

Proof. If $|Y| = 1$, then for $A \subset X$ and $A \notin \mathcal{F}$, $\mathcal{F}_Y \cup A$ is not a filter, as $Y \not\subset A$ so any $B \in \mathcal{F}_Y$ would have $A \cap B = \emptyset$. If $|Y| \neq 1$, then take any element $y \in Y$ and add all subset of X containing y to $\mathcal{Y}$, then we obtain a filter containing $\mathcal{F}_Y$, so $\mathcal{F}_Y$ is not an ultrafilter. □

Proposition 2.2. *For a filter $\mathcal{F}$ on X, the following are equivalent:*

- *For any $Y \subset X$, exactly one of Y or $X \setminus Y$ is in $\mathcal{F}$.*

- $\mathcal{F}$ *is an ultrafilter.*

Proof. $\Rightarrow$: If $\mathcal{F}$ is not an ultrafilter, then it is contained in another filter $\mathcal{F}'$ and there is a set $A \subset X$ such that $A \in \mathcal{F}' \setminus \mathcal{F}$. Then $X \setminus A$ is also not in $\mathcal{F}$, because otherwise, $X \setminus A \in \mathcal{F}'$ and $\emptyset \in \mathcal{F}'$, which is impossible. Therefore we conclude that $\mathcal{F}$ is an ultrafilter.

$\Leftarrow$: If $\mathcal{F}$ is an ultrafilter, then $\mathcal{F}$ must contain at least one of Y or $X \setminus Y$. Otherwise, let $\mathcal{F}' = \mathcal{F} \cup \mathcal{F}_Y$ where $\mathcal{F}_Y$ is the principal filter $A|Y \subset A$. $\mathcal{F}'$ is also a filter, since if we take any two sets A_1 and A_2 in $\mathcal{F}'$. Consider the non-trivial cases. Suppose $A_1 \in \mathcal{F}$ and $A_2 \in \mathcal{F}_Y$, we claim that $A_1 \cap A_2 \neq \emptyset$. $Y \subset A_2$, and $A_1 \cap Y \neq \emptyset$ too, because otherwise, $A_1 \subset X \setminus Y$ and by the definition of filters, $X \setminus Y \in \mathcal{F}$. $\square$