Homework 1

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1.

Proof. Let $z = x+iy$, $z_1 = a+ib$ and $z_2 = c+id$. $|z-z_1| = (x-a)^2+(y-b)^2$, $|z-z_2| = (x-c)^2+(y-d)^2$,

$$\begin{aligned}
0 &= |z-z_1|^2 - A^2|a-z_2|^2 \\
&= (x-a)^2 + (y-b)^2 - A^2(x-c)^2 - A^2(y-d)^2 \\
&= (1-A^2)x^2 + (1-A^2)y^2 + 2(A^2c-a)x + 2(A^2d-b)y + a^2 + b^2 - (Ac)^2 - (Ad)^2 \\
&= \left(x - \frac{a-A^2c}{1-A^2}\right)^2 + \left(y - \frac{b-A^2d}{1-A^2}\right)^2 - \frac{A^2}{(1-A^2)^2}[(a-c)^2 + (b-d)^2]
\end{aligned}$$

So if $A > 0$ and $A \neq 1$ and $z_1 \neq z_2$, then $\{z \in \mathbb{C} : |z-z_1| = A|z-z_2|\}$ is a circle with center at $((a-A^2c)/(1-A^2), (b-A^2d)/(1-A^2))$ and radius $A/(1-A^2)(|z_1-z_2|)$ $\square$

2.

Proof. First prove that the identity is invariant under scalings, translations and rotations. If we multiple z_1, z_2, z_3 by $\alpha \in \mathbb{R}$, we have

$$\alpha^2(z_1^2 + \alpha^2(z_2^2 + \alpha^2(z_3^2 = \alpha^2 z_1 z_2 + \alpha^2 z_2 z_3 + \alpha^2 z_3 z_1$$

This doesn't change the identity. Second, if we rotate z_i by the same angle, such that $z_i' = r_i e^{i(\theta_i+\phi)}$ where $z_i = r_i e^{i\theta_i}$, we have

$$\begin{aligned}
& r_1^2 e^{2i(\theta_1+\phi)} + r_2^2 e^{2i(\theta_2+\phi)} + r_3^2 e^{2i(\theta_3+\phi)} \\
= \;& e^{2i\phi}(r_1^2 e^{2i\theta_1} + r_2^2 e^{2i\theta_2} + r_3^2 e^{2i\theta_3}) \\
= \;& e^{2i\phi}(r_1 r_2 e^{i(\theta_1+\theta_2)} + r_2 r_3 e^{i(\theta_2+\theta_3)} + r_3 r_1 e^{i(\theta_3+\theta_1)}) \\
= \;& r_1 r_2 e^{i(\theta_1+\theta_2+2\phi)} + r_2 r_3 e^{i(\theta_2+\theta_3+2\phi)} + r_3 r_1 e^{i(\theta_3+\theta_1+2\phi)}
\end{aligned}$$

Lastly, if we subtract z from $z_i, i = 1, 2, 3$, we have

$$\begin{aligned} & (z_1 - z)^2 + (z_2 - z)^2 + (z_3 - z)^2 \\ = \; & z_1^2 - 2z_1 z + z^2 + z_2^2 - 2z_2 z + z^2 + z_3^2 - 2z_3 z + z^2 \\ = \; & z_1 z_2 + z_2 z_3 + z_3 z_1 - 2(z_1 + z_2 + z_3)z + 3z^2 \\ = \; & (z_1 - z)(z_2 - z) + (z_2 - z)(z_3 - z) + (z_3 - z)(z_1 - z) \end{aligned}$$

If z_1, z_2, z_3 are vertices of a equilateral triangle, we translate the triangle such that $z_1 = 1$, $z_2 = 0$ and $z_3 = 1/2 + i\sqrt(3)/2$.

$$\begin{aligned} & (z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 \\ = \; & 1 + (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)^2 + (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^2 \\ = \; & 0 \end{aligned}$$

so $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$.

On the other hand, if $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$, we want to show that $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$. We can translate, rotate and scale z_1, z_2, z_3 such that $z_1 = 0$, $z_2 = 1$. Let $z_2 - z_3 = \omega$, then $z_3 - z_1 = -(1 + \omega)$. Now want to show $|\omega| = |1 + \omega| = 1$.

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$$0 + 1 + (1 + \omega)^2 = 0 - (1 + \omega)1 + \omega + \omega^2 = 0$$

so $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{1}$ or $\omega = -\frac{1}{2} - i\frac{\sqrt{3}}{1}$, and $|\omega| = |1 + \omega| = 1$. □

3.

- $z^4 = -1$, $z^2 = \pm i$, $z = \pm\sqrt{i} or z = \pm i\sqrt{i}$.
- not solved yet
- $(z^3 + 1)^2 = -1$, $z^3 + 1 = \pm i$, $z^3 = -1 \pm i$, $z = \sqrt[3]{-1 \pm i}$

4.

Proof. Want to prove that

$$\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = \frac{|z_1 - z_2|}{|1 - \bar{z}_1 z_2|} < 1$$

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\bar{z}_1 z_2 < |1 - \bar{z}_1 z_2|^2 = 1 - 2\bar{z}_1 z_2 + |z_1|^2|z_2|^2$$
$$|z_1|^2 + |z_2|^2 < 1 + |z_1|^2|z_2|^2$$
$$|z_1|^2(1 - |z_2|^2) + |z_2|^2 < 1 \tag{1}$$

Let $\lambda = |z_2|^2 \in (0, 1]$. When $\lambda = 1$, Eq.(1) holds. When $\lambda \in (0, 1)$ and if $|z_1|^2(1 - \lambda) + \lambda \geq 1$, then $|z_1|^2 \geq 1$, a contradiction.

6.

Proof.

- $\prod_{j=0}^{n-1}(z - e^{\frac{2\pi i j}{n}}) = (z - 1)\prod_{j=1}^{n-1}(z - e^{\frac{2\pi i j}{n}}) = z^n - 1 = (z - 1)$ □

□