THE ARZELA–ASCOLI THEOREM

Let Ω be a region in $\mathbb{C}$. Let $\Omega_{\mathbb{Q}}$ denote its subset of points with rational coordinates,

$$\Omega_{\mathbb{Q}} = \{x + iy \in \Omega : x, y \in \mathbb{Q}\}.$$

This subset is useful because it is small in the sense that is countable, but large in the sense that it is dense in Ω.

Definition 0.1. *A family $\mathcal{F}$ of complex-valued functions on Ω is* pointwise bounded *if*

$$\textit{for each } z \in \Omega, \quad \sup_{f \in \mathcal{F}} \{|f(z)|\} < \infty.$$

This does not imply that any $f \in \mathcal{F}$ is bounded on Ω, as demonstrated by the family

$$\mathcal{F} = \{f_n : n \in \mathbb{Z}^+\} \quad \text{where } f_n(z) = z/n \text{ for } z \in \mathbb{C}.$$

Nor is it implied if every $f \in \mathcal{F}$ is bounded on Ω, as demonstrated by the family

$$\mathcal{F} = \{f_n : n \in \mathbb{Z}^+\} \quad \text{where } f_n(z) = n \text{ for } z \in \mathbb{C}.$$

Definition 0.2. *A family $\mathcal{F}$ of complex-valued functions on Ω is* equicontinous *if for every $\varepsilon > 0$ and $z \in \Omega$, there exists some $\delta > 0$ such that for all $\tilde{z} \in \Omega$,*

$$|\tilde{z} - z| < \delta \implies |f(\tilde{z}) - f(z)| < \varepsilon \textit{ for all } f \in \mathcal{F}.$$

The idea here is that each $f \in \mathcal{F}$ is pointwise continuous on Ω, and at each point $z \in \Omega$, given $\varepsilon > 0$, the same $\delta > 0$ works simultaneously for all $f \in \mathcal{F}$ in the definition of continuity at z.

Theorem 0.3 (Arzela–Ascoli)**.** *Let Ω be a region in $\mathbb{C}$, and let $\mathcal{F}$ be a pointwise bounded, equicontinuous family of complex-valued functions on Ω. Then every sequence $\{f_n\}$ in $\mathcal{F}$ has a convergent subsequence, the convergence being uniform on compact subsets.*

Proof. Let $\{f_n\}$ be a sequence in $\mathcal{F}$.

First we use the given pointwise boundedness to prove that $\{f_n\}$ has a subsequence that converges on $\Omega_{\mathbb{Q}}$. The idea is a variant of Cantor's diagonal argument. Since $\Omega_{\mathbb{Q}}$ is countable, write

$$\Omega_{\mathbb{Q}} = \{z_1, z_2, z_3, \dots\}.$$

The complex sequence

$$\{f_1(z_1), f_2(z_1), f_3(z_1), \dots\}$$

is bounded, and so it contains a convergent subsequence. Relabel the convergent subsequence as follows:

$$\{f_{1,1}(z_1), f_{1,2}(z_1), f_{1,3}(z_1), \dots\} \text{ converges.}$$

Next, the complex sequence

$$\{f_{1,1}(z_2), f_{1,2}(z_2), f_{1,3}(z_2), \dots\}$$

is again bounded, so it too contains a convergent subsequence. Relabel it:

$$\{f_{2,1}(z_2), f_{2,2}(z_2), f_{2,3}(z_2), \dots\} \text{ converges.}$$

And the complex sequence

$$\{f_{2,1}(z_3), f_{2,2}(z_3), f_{2,3}(z_3), \dots\}$$

is bounded, so it contains a convergent subsequence:

$$\{f_{3,1}(z_3), f_{3,2}(z_3), f_{3,3}(z_3), \dots\} \text{ converges.}$$

Continuing this process gives rise to an array,

$$\begin{matrix} f_{1,1} & f_{1,2} & f_{1,3} & \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} & \cdots \\ f_{3,1} & f_{3,2} & f_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

The first row is a sequence of functions that converges at z_1. The second row is a subsequence of the first row, and it converges at z_1 and at z_2. The third row is a subsequence of the second row, and it converges at z_1, at z_2, and at z_3. And so on. Consider the sequence down the diagonal,

$$\{f_{1,1}, f_{2,2}, f_{3,3}, \dots\}.$$

This is a subsequence of the original sequence $\{f_n\}$, and it converges at each $z \in \Omega_{\mathbb{Q}}$. After relabeling, we may assume that the original sequence $\{f_n\}$ converges on $\Omega_{\mathbb{Q}}$.

Next we use the given equicontinuity to prove that in fact $\{f_n\}$ converges on all of Ω. This is a typical three-epsilon argument. Given any $z \in \Omega$ and any $\varepsilon > 0$, consider the $\delta > 0$ provided by equicontinuity. Since $\Omega_{\mathbb{Q}}$ is dense in Ω, there exists a point $z_{\mathbb{Q}} \in \Omega_{\mathbb{Q}}$ such that

$$|z_{\mathbb{Q}} - z| < \delta.$$

Since the complex sequence $\{f_n(z_{\mathbb{Q}})\}$ converges, it is Cauchy, meaning that there exists a starting index N such that for all integers n and m,

$$n, m > N \implies |f_n(z_{\mathbb{Q}}) - f_m(z_{\mathbb{Q}})| < \varepsilon.$$

Consequently, the complex sequence $\{f_n(z)\}$ is Cauchy as well,

$$\begin{aligned} n, m > N \implies |f_n(z) - f_m(z)| &\le |f_n(z) - f_n(z_{\mathbb{Q}})| \\ &\quad + |f_n(z_{\mathbb{Q}}) - f_m(z_{\mathbb{Q}})| \\ &\quad + |f_m(z_{\mathbb{Q}}) - f_m(z)| \\ &< 3\varepsilon. \end{aligned}$$

Since the complex sequence $\{f_n(z)\}$ is Cauchy, it converges.

Third we prove that the pointwise limit function

$$g = \lim_n f_n : \Omega \longrightarrow \mathbb{C}$$

is continuous. Let $\varepsilon > 0$ be given and let $z \in \Omega$ be given. The equicontinuity of $\mathcal{F}$ supplies a corresponding $\delta = \delta_z(\varepsilon, \mathcal{F}) > 0$. For any $\tilde{z} \in \Omega$ such that $|\tilde{z} - z| < \delta$ and for any $n \in \mathbb{Z}^+$,

$$|g(\tilde{z}) - g(z)| \le |g(\tilde{z}) - f_n(\tilde{z})| + |f_n(\tilde{z}) - f_n(z)| + |f_n(z) - g(z)|.$$

By equicontinuity, the middle term is less than ε for any n. By the pointwise convergence of $\{f_n\}$ to g, for some starting index $N = N(z, \tilde{z})$ the first and last terms are less than ε for all $n > N$. That is, for all $\tilde{z} \in \Omega$ such that $|\tilde{z} - z| < \delta$,

$$|g(\tilde{z}) - g(z)| < 3\varepsilon \quad \text{for all } n > N.$$

But $g(\tilde{z}) - g(z)$ is independent of n, so the "for all $n > N$" in the display is irrelevant, and g is continuous at z. The equicontinuity of $\mathcal{F}$ and the continuity of g combine to show that also $\mathcal{F} \cup \{g\}$ is equicontinuous.

Finally we prove that the convergence of $\{f_n\}$ to g is uniform on compact subsets of Ω. Let K be such a compact set, and let $\varepsilon > 0$ be given. We need a starting index N such that for all integers n,

$$n > N \implies |f_n(z) - g(z)| < 3\varepsilon \text{ for all } z \in K.$$

For each $z \in K$ there exists some $\delta_z = \delta_z(\varepsilon, \mathcal{F} \cup \{g\}) > 0$ such that for all $\tilde{z} \in K$,

$$|\tilde{z} - z| < \delta_z \implies \begin{cases} |f_n(\tilde{z}) - f_n(z)| < \varepsilon \text{ for all } n \in \mathbb{Z}^+ \\ |g(z) - g(\tilde{z})| < \varepsilon. \end{cases}$$

And because $\{f_n(z)\}$ converges to $g(z)$, there exists some $N_z \in \mathbb{Z}^+$ such that for all integers n,

$$n > N_z \implies |f_n(z) - g(z)| < \varepsilon.$$

So, for all $\tilde{z} \in K$ and all integers n,

$$\left\{ \begin{matrix} |\tilde{z} - z| < \delta_z, \\ n > N_z \end{matrix} \right\} \implies |f_n(\tilde{z}) - g(\tilde{z})| \leq \begin{pmatrix} |f_n(\tilde{z}) - f_n(z)| \\ +|f_n(z) - g(z)| \\ +|g(z) - g(\tilde{z})| \end{pmatrix} < 3\varepsilon.$$

This shows that the sequence $\{f_n\}$ converges uniformly on $B(z, \delta_z) \cap K$. So consider an open cover of the compact set K,

$$K = \bigcup_{z \in K} B(z, \delta_z) \cap K.$$

By compactness, there exists a finite subcover,

$$K = \bigcup_{j=1}^{k} B(z_j, \delta_{z_j}) \cap K.$$

Define

$$N = \max(N_{z_1}, \ldots, N_{z_k}\}.$$

Then for all integers n and m, the desired condition holds,

$$n > N \implies |f_n(z) - g(z)| < 3\varepsilon \text{ for all } z \in K.$$

This completes the proof. □